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Alexander G. Nikolaev, Sheldon H. Jacobson,

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TECHNICAL NOTE

Stochastic Sequential Decision-Making with a Random Number of Jobs

Alexander G. Nikolaev
Department of Industrial Engineering and Management Sciences, Northwestern University, Evanston, Illinois 60201,
a-nikolaev@northwestern.edu

Sheldon H. Jacobson
Department of Computer Science, University of Illinois at Urbana–Champaign, Urbana, Illinois 61801,
shj@illinois.edu

This paper addresses a class of problems in which available resources need to be optimally allocated to a random number of jobs with stochastic parameters. Optimal policies are presented for variations of the sequential stochastic assignment problem and the dynamic stochastic knapsack problem, in which the number of arriving jobs is unknown until after the final arrival, and the job parameters are assumed to be independent but not identically distributed random variables.

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1. Introduction

Sequential resource allocation problems with uncertainty have received much attention in the literature. This paper focuses on two problems in this field: the sequential stochastic assignment problem (SSAP), and the dynamic stochastic knapsack problem (DSKP).

Derman et al. (1972) introduced the SSAP: Given a known finite number of jobs with independent and identically distributed (i.i.d.) reward values that arrive sequentially, one at a time, how should these jobs be assigned to workers with known finite success rates, where the assignment of each job should be determined nonanticipatively at the time the job arrives? Derman et al. (1972) establish an optimal policy that maximizes the total expected reward, where the reward is the sum of products of job values and worker success rates over all assignments.

Theoretical extensions to the investigation by Derman et al. (1972) include scenarios in which various continuous distributions of job arrival times are considered (Albright 1974, Sakaguchi 1972, Righter 1987). Other variations and applications of SSAP have been addressed by Derman et al. (1975), Nakai (1986), Su and Zenios (2005), Nikolaev et al. (2007), and McLay et al. (2009). Kennedy (1986) established the most general result for SSAP by removing the assumption of independence and proving that threshold policies are optimal for any problem of this type, although the thresholds that define such policies may be random variables and difficult to compute.

DSKP was first defined by Papastavrou et al. (1996): Given a limited fixed resource capacity, and jobs with i.i.d. weights and reward values that arrive sequentially, one at a time, how should the available resource be allocated by nonanticipatively accepting or rejecting jobs? Papastavrou et al. (1996) analyze the case of DSKP formulated for a time horizon of a given number of discrete periods, with a fixed constant probability of a job arrival in each such period, for different forms of a joint probability distribution function for job weights and values. Kleywegt and Papastavrou (1998, 2001) consider Poisson arrivals in DSKP. Other variations and applications of DSKP have been discussed by Prastakos (1983), Lu et al. (1999), and Van Slyke and Young (2000).

This paper uses conditioning arguments and the results from Kennedy (1986) to consider extensions to SSAP and DSKP, where the number of jobs is unknown until after the final arrival, but follows a (given) discrete distribution that has either finite or infinite support. The arriving jobs are assumed to be independent but not necessarily identically distributed. Note that an optimal policy for a more restricted version of SSAP was presented by Sakaguchi (1983) in a different form and without a formal proof; this paper formally proves and extends those results. Also, the original finite-horizon formulation of DSKP (Papastavrou et al. 1996) considers discrete arrivals with a random number of jobs that follows a binomial distribution; this paper generalizes these results to include other discrete distributions.

The paper is organized as follows. Section 2 shows how the results by Kennedy (1986) can be used to address the SSAP extension. Section 3 presents a dynamic programming (DP) algorithm to solve the DSKP extension. Section 4 offers concluding comments.

2. SSAP with a Random Number of Jobs

This section addresses the SSAP with a random number of jobs. First, the case in which the distribution of the number of jobs has finite support is considered. This result is then extended to the infinite support case.

2.1. Finite Case

The base problem (BP) is formally stated.

Given. $M \in \mathcal{Z}^+$ workers available to perform N jobs; a fixed success rate p_w associated with worker $w = 1, 2, \ldots, M$; probability mass function (pmf) P_n for the number of jobs with independent values arriving sequentially, one at a time; for each job $j = 1, 2, \ldots, N$, a job value cumulative distribution function (cdf) $F_j(x_j)$.

Objective. Find a policy π^* that determines the assignment of jobs to workers, $A_{wj} \in \{0, 1\}, w = 1, 2, \ldots, M$, $j = 1, 2, \ldots, N$, such that $\sum_{j=1}^{N} A_{wj} \leqslant 1$, $w = 1, 2, \ldots, M$, $\sum_{w=1}^{M} A_{wj} \leqslant 1$, $j = 1, 2, \ldots, N$, and $E^{\pi^*}_{P_n, \{F_j\}_{j=1}^N}[\sum_{w=1}^{M}\sum_{j=1}^{N} p_w A_{wj} X_j]$ is maximized.

The main challenge presented by BP is the randomness in the *number* of arriving jobs. To address this challenge, an auxiliary problem (AP) can be created where the number of jobs is fixed, but the job values are dependent. Using BP, the AP is created as follows. Fix the number of workers at $N_{\max}$, the largest value that N can take on (i.e., $\sum_{n=0}^{N_{\max}} P_n = 1$). If $N_{\max} \leqslant M$, set $p'_i = p_i$ for $i = 1, 2, \ldots, N_{\max}$. If $N_{\max} > M$, set $p'_i = p_i$ for $i = 1, 2, \ldots, M$ and $p'_{M+1} = p'_{M+2} = \cdots = p'_{N_{\max}} = 0$. Also, let $X'_1 = X_1$, and for any $j = 2, \ldots, N_{\max}$, set

$$X'_j = \begin{cases} X_j \text{ with probability } \dfrac{\sum_{i=j}^{N_{\max}} P_i}{\sum_{i=j-1}^{N_{\max}} P_i}, & \text{if } X'_{j-1} > 0 \\ 0 \text{ with probability } \dfrac{P_{j-1}}{\sum_{i=j-1}^{N_{\max}} P_i}, & \text{if } X'_{j-1} > 0 \\ 0, & \text{if } X'_{j-1} = 0. \end{cases} \quad (1)$$

The AP is now formally stated.

Given. $N_{\max}$ workers available to perform $N_{\max}$ jobs; a fixed success rate p'_w associated with worker $w = 1, 2, \ldots, N_{\max}$; $N_{\max}$ jobs with independent values arriving sequentially, one at a time; for each job $j = 1, 2, \ldots, N_{\max}$, a job value cdf $F'_j(x'_j)$.

Objective. Find a policy π'^* that determines the assignment of jobs to workers, $A'_{wj} \in \{0, 1\}$, $w = 1, 2, \ldots, N_{\max}$, $j = 1, 2, \ldots, N_{\max}$, such that $\sum_{j=1}^{N_{\max}} A'_{wj} \leqslant 1$, $w = 1, 2, \ldots, N_{\max}$, $\sum_{w=1}^{M} A'_{wj} \leqslant 1$, $j = 1, 2, \ldots, N_{\max}$, and $E^{\pi'^*}_{\{F'_j\}_{j=1}^{N_{\max}}}[\sum_{w=1}^{N_{\max}}\sum_{j=1}^{N_{\max}} p'_w A'_{wj} X'_j]$ is maximized.

By design, BP and AP are closely related. Because $X_j > 0$ for any $j = 1, 2, \ldots, N$, then by (1), the first N jobs in BP and AP have the same values. Also, the values of the subsequent jobs $j = N+1, N+2, \ldots, N_{\max}$ in AP are equal to zero, and no additional reward can be earned (i.e., event {Job "j" does not arrive in BP} $\equiv \{X'_j = 0$ in AP$\}$). Theorem 1 establishes that if an optimal policy for AP is available, then an optimal policy for BP can be obtained.

THEOREM 1. *Let π'^* be an optimal policy for AP. Then, an optimal policy for BP, π^*, is obtained using rules*: (1) *whenever a job arrives and π'^* assigns a worker with success rate zero, discard the job, and* (2) *whenever a job arrives and π'^* assigns a worker with success rate $p > 0$, assign a worker with the same success rate.*

PROOF. See e-companion. An electronic companion to this paper is available as part of the online version that can be found at http://or.journal.informs.org/.

To determine an optimal policy π'^* for AP, the result by Kennedy (1986) can be applied. Using the notations introduced for AP, let job values X'_j, $j = 1, 2, \ldots, N_{\max}$, be any (not necessarily i.i.d.) random variables. For any $n = 1, 2, \ldots, N_{\max}$ and $m = 0, 1, \ldots, N_{\max}$, define random variables $Z^{N_{\max}}_{m,n}$ such that (1) $Z^{N_{\max}}_{0,n} \equiv +\infty$, for $1 \leqslant n \leqslant N_{\max}$; (2) $Z^{N_{\max}}_{m,n} \equiv -\infty$, for $m > N_{\max} - n + 1$; (3) $Z^{N_{\max}}_{1,N_{\max}} = X'_{N_{\max}}$; and (4) $Z^{N_{\max}}_{m,n} = [X'_n \vee \mathrm{E}[Z^{N_{\max}}_{m,n+1} | \mathcal{F}_n]] \wedge \mathrm{E}[Z^{N_{\max}}_{m-1,n+1} | \mathcal{F}_n]$, for $1 \leqslant m \leqslant N_{\max} - n + 1$, $n \leqslant N_{\max} - 1$, where $\mathcal{F}_n$, $n = 1, 2, \ldots, N_{\max} - 1$, is a sigma-field over all possible realizations of vector $\{X'_i\}_{i=1}^n$, $n = 1, 2, \ldots, N_{\max} - 1$, $\vee$ denotes the maximum, and $\wedge$ denotes the minimum.

For any $n = 1, 2, \ldots, N_{\max}$ and $m = 1, \ldots, N_{\max}$, the random variable $Z^{N_{\max}}_{m,n}$ represents the expected value of a job to which the mth most skilled (mth best) worker is expected to be assigned upon the arrival and assignment of job n. At the time when job n with value x'_n arrives, the following hold:

- If job n is assigned to the mth best worker, then the value of $Z^{N_{\max}}_{m,n}$ is equal to x'_n.
- If job n is assigned to a more skilled worker than the mth best, then the mth best worker becomes the $(m-1)$th best, and $Z^{N_{\max}}_{m,n}$ is equal to $\mathrm{E}[Z^{N_{\max}}_{m-1,n+1} | \mathcal{F}_n]$.
- If job n is assigned to a less skilled worker than the mth best, then the mth best worker remains the mth best, and $Z^{N_{\max}}_{m,n}$ is equal to $\mathrm{E}[Z^{N_{\max}}_{m,n+1} | \mathcal{F}_n]$.

Theorem 2 shows that it makes sense to assign job n to a more skilled worker than the mth best only if x'_n is greater than the value of $\mathrm{E}[Z^{N_{\max}}_{m-1,n+1} \mid \mathcal{F}_n]$, and to assign job n to a less skilled worker than the mth best only if x'_n is less than the value of $\mathrm{E}[Z^{N_{\max}}_{m,n+1} \mid \mathcal{F}_n]$.

THEOREM 2 (KENNEDY 1986). *Whenever job $n = 1, 2, \ldots, N_{\max} - 1$ arrives, the line segment $(-\infty, +\infty) \subset \mathcal{R}$ is partitioned into $N_{\max} - n + 1$ random intervals defined by the breakpoints $+\infty \geqslant \mathrm{E}[Z^{N_{\max}}_{1,n+1} \mid \mathcal{F}_n] \geqslant \mathrm{E}[Z^{N_{\max}}_{2,n+1} \mid \mathcal{F}_n] \geqslant \cdots \geqslant \mathrm{E}[Z^{N_{\max}}_{N_{\max}-n,n+1} \mid \mathcal{F}_n] \geqslant -\infty$. Then, the optimal assignment policy is to assign the nth job to the worker with the mth highest success rate* (*available at the time of the assignment*) *if x'_n lies in the mth highest of these intervals, or, equivalently, if $Z^{N_{\max}}_{m,n} = x'_n$.*

Theorem 2 establishes the form of an optimal policy for any problem, where the objective function is given as the expectation of a summation of products. However, this result has seen limited use because finding the conditional expectations of recursively defined random variables $Z^{N_{\max}}_{m,n}$, $1 \leqslant m \leqslant N_{\max} - n + 1$, $n \leqslant N_{\max} - 1$, is computationally intractable in many cases, especially when X'_j, $j = 1, 2, \ldots, N_{\max}$ are dependent. For any $n = 1, 2, \ldots, N_{\max} - 1$, conditioning on the sigma-field $\mathcal{F}_n$ implies that the interval breakpoints depend on a sequence of values of jobs 1 through n, and hence, for any such sequence, the breakpoint values may be different. However, if the nature of the dependency is as defined in (1), then a closed form optimal assignment policy for AP can be obtained.

Theorem 3. *Whenever job $n = 1, 2, \ldots, N_{\max} - 1$ arrives in AP, the optimal assignment policy is to assign the nth job to the worker with the mth highest success rate (available at the time the assignment decision has to be made) if x'_n lies in the mth highest of the intervals, defined by the fixed breakpoints*

$$\mathrm{E}[Z^{N_{\max}}_{1,n+1} | X'_n > 0],$$
$$\mathrm{E}[Z^{N_{\max}}_{2,n+1} | X'_n > 0], \ldots, \mathrm{E}[Z^{N_{\max}}_{N_{\max}-n,n+1} | X'_n > 0].$$

These breakpoints are computed recursively:

$$\begin{aligned}&\mathrm{E}[Z^{N_{\max}}_{m,n+1} | X'_n > 0]\\ &\quad= \frac{\sum_{i=n+1}^{N_{\max}} P(N=i)}{\sum_{i=n}^{N_{\max}} P(N=i)} \Big(F_{n+1}(\mathrm{E}[Z^{N_{\max}}_{m,n+2} | X'_{n+1} > 0])\\ &\qquad \cdot \mathrm{E}[Z^{N_{\max}}_{m,n+2} | X'_{n+1} > 0] + \int_{\mathrm{E}[Z^{N_{\max}}_{m,n+2}|X'_{n+1}>0]}^{\mathrm{E}[Z^{N_{\max}}_{m-1,n+1}|X'_{n+1}>0]} x\, dF_{n+1}(x)\\ &\qquad + (1 - F_{n+1}(\mathrm{E}[Z^{N_{\max}}_{m-1,n+1} | X'_{n+1} > 0]))\\ &\qquad \cdot \mathrm{E}[Z^{N_{\max}}_{m-1,n+1} | X'_{n+1} > 0]\Big). \qquad (2)\end{aligned}$$

Proof. See e-companion.

The backward recursion (2) begins with the last ($N_{\max}$)th job, for which the breakpoints are 0, $+\infty$ (therefore, the job is assigned to the best remaining worker available). Next, the breakpoints for job ($N_{\max} - 1$) are 0, $P_{N_{\max}}/(P_{N_{\max}-1} + P_{N_{\max}})\mathrm{E}[X_{N_{\max}}]$, $+\infty$. To compute the breakpoints for all $N_{\max}$ jobs, proceed in the same manner, down to job 1.

2.2. Infinite Case

The results of Theorems 1 and 3 can be extended to the case in which the pmf for the number of jobs in BP has infinite support. In this case, the proof of Theorem 1 is unchanged. The rewards earned in BP and AP by making assignments for any pair of sequences s and s' (see Theorem 1), respectively, remain the same, because every such sequence has only a finite number of jobs. Therefore, solving AP, where the pmf of the number of jobs has infinite support, solves BP. Kennedy (1986) establishes the form of an optimal policy for such problems, as summarized in Theorem 4.

Theorem 4 (Kennedy 1986). *Assume that* $\mathrm{E}[\sup_n |X'_n|] < +\infty$, *and* $\lim_{n\to+\infty} X'_n = 0$. *Then, an infinite sequence* $\{Z^{N_{\max}}_{m,n}\}^{+\infty}_{N_{\max}=1}$ *converges to a finite limit*

$$Z_{m,n} \equiv \lim_{N_{\max}\to+\infty} Z^{N_{\max}}_{m,n},$$

and Theorem 2 holds with the breakpoints expressed as $+\infty$, $\mathrm{E}[Z_{1,n+1} | \mathcal{F}_n]$, $\mathrm{E}[Z_{2,n+1} | \mathcal{F}_n], \ldots, -\infty$.

Theorem 4 establishes that finding an optimal policy for AP (and, using Theorem 1, BP), where the pmf for the number of jobs has infinite support, can be approached by considering a sequence of AP's with fixed (bounded) $N_{\max}$, and letting $N_{\max} \to +\infty$. Note that the distributions of job values in such APs (see (1)) depend on the pmfs of the number of jobs, and hence it is necessary to define the pmf $P^{N_{\max}}$ of the number of jobs for each AP with $N_{\max} = 1, 2, \ldots$. To match the set-up described in Kennedy (1986), the distribution of the value of job $j = 1, 2, \ldots$ has to be the same in each of those APs with $N_{\max} = 1, 2, \ldots$. To satisfy this requirement, set $P^{N_{\max}}_i = P_i / \sum_{k=1}^{N_{\max}} P_k$ for any $i = 1, 2, \ldots, N_{\max}$, $N_{\max} = 1, 2, \ldots$.

2.3. Illustrative Example

Theorems 3 and 4 describe the necessary computations involved in deriving optimal policies for SSAP with a random number of jobs. An example is provided to illustrate how these computations are performed.

Example. Given $M = 4$. $P_1 = P_2 = P_3 = P_4 = 1/4$. $F_j(x) = x$, $0 \leqslant x \leqslant 1$, for $j = 1, 2, 3, 4$.

For this example, $N_{\max} = 4$. Define $b^{N_{\max}}_{m,n+1} = \mathrm{E}[Z^{N_{\max}}_{m,n+1} | X'_n > 0]$ for $1 \leqslant m \leqslant N_{\max} - n + 1$ and $1 \leqslant n \leqslant N_{\max} - 1$. By (2),

$$n = 3:\ b^4_{1,4} = \frac{\sum_{i=4}^4 P_i}{\sum_{i=3}^4 P_i} \int_{-\infty}^{+\infty} x\, dF_4(x) = \frac{1/4}{2/4} \cdot \tfrac{1}{2} = \frac{1}{4};$$

$$\begin{aligned}n = 2:\ b^4_{2,3} &= \frac{\sum_{i=3}^4 P_i}{\sum_{i=2}^4 P_i} \left(\int_{-\infty}^{1/4} x\, dF_3(x) + (1 - F_3(b^4_{1,4})) b^4_{1,4} \right)\\ &= \frac{2}{3} \cdot \left(\frac{1}{32} + \frac{3}{4} \cdot \frac{1}{4} \right) = \frac{7}{48},\end{aligned}$$

$$b^4_{1,3} = \frac{\sum_{i=3}^4 P_i}{\sum_{i=2}^4 P_i} \left(F_3(b^4_{1,4}) b^4_{1,4} + \int_{-\infty}^{1/4} x\, dF_3(x) \right) = \frac{17}{48};$$

$n = 1$: $b^4_{3,2} \approx 0.1014$, $b^4_{2,2} \approx 0.2266$, $b^4_{1,2} \approx 0.422$. □

The derived optimal policy can be compared with the policy that would be optimal if the number of jobs was not random. According to an optimal policy for SSAP (Derman et al. 1972) with $N = 4$, with job values distributed as in the example, the interval breakpoints that determine the assignments for the third, second, and first arriving jobs (respectively) would be

$$n=3\colon\ a^4_{1,4}=0.5;\quad n=2\colon\ a^4_{2,3}=\frac{18}{48},\ a^4_{1,3}=\frac{30}{48};$$
$$n=1\colon\ a^4_{3,2}\approx 0.3047,\ a^4_{2,2}=0.5,\ a^4_{1,2}\approx 0.6953.$$

The interval breakpoint values obtained in the example's solution are smaller, which means that workers with higher success rates are used earlier than in the solution to the respective instance of SSAP with the known number of arrivals, at all assignment stages.

3. DSKP with a Random Number of Jobs

This section analyzes the DSKP with a random number of jobs and presents a dynamic program that leads to the derivation of an optimal assignment policy. The DSKP is formally stated.

Given. Resource of capacity C available for allocation to N jobs; pmf P_n for the number of jobs with independent weights and values arriving sequentially, one at a time; for each job $j=1,2,\ldots,N$, a joint cdf $F_j(w,x)$ for the job weight and value.

Objective. Find a policy π^* that determines the assignments, $A_j\in\{0,1\}$, $j=1,2,\ldots,N$, such that $\sum_{j=1}^N A_j\leqslant 1$, $\sum_{j=1}^N A_j w_j\leqslant C$, and $E^{\pi*}_{P_n,\{F_j\}_{j=1}^N}[\sum_{j=1}^N A_j X_j]$ is maximized.

For any $j=1,2,\ldots$ and $c\in[0,C]$, let V_j^c denote the optimal accumulated reward from the allocation of resource capacity c to jobs $j,j+1,\ldots,N$, and let EV_j^c denote the optimal *conditional* expected accumulated reward from the allocation of resource capacity c to jobs $j,j+1,\ldots,N$, given that job $j-1$ has arrived. By definition, $EV_1^C=E^{\pi*}_{P_n,\{F_j\}_{j=1}^N}[\sum_{j=1}^N A_j X_j]$. Theorem 5 establishes an assignment policy that guarantees the optimal expected resource allocation.

Theorem 5. *Suppose that the remaining resource capacity is c, and job j with weight w_j and value x_j arrives. Then, it is optimal to set*

$$A_j=\begin{cases}1 & \text{if } x_j+EV_{j+1}^{c-w_j}\geqslant EV_{j+1}^c \text{ and } w_j\leqslant c\\ 0 & \text{if } x_j+EV_{j+1}^{c-w_j}< EV_{j+1}^c \text{ or } w_j> c.\end{cases}\tag{3}$$

Note that the quantity $x_j+EV_{j+1}^{c-w_j}$ depends on the parameters (weight and value) of job j. These parameters are known at the time the assignment decision for job j is to be made. Therefore, each optimal assignment decision, described by (3), is determined by EV_{j+1}^c and $EV_{j+1}^{c-w_j}$.

Theorem 5 follows from the fundamental argument of DP: Each assignment must maximize the sum of an immediate reward and the expected future reward. Note that rule (3) is of the same form as in Papastavrou et al. (1996), except that EV_j^c, $j=1,2,\ldots$, $c\in[0,C]$, are *conditional*. This allows one to include the consideration of the pmf of N into the DP formulation and hence determine the optimal allocation policy for the case with a random number of jobs.

The expected values EV_j^c, $j=1,2,\ldots$, $c\in[0,C]$ can be computed using a DP recursion. However, the recursion and its boundary conditions depend on the number of arriving jobs, which is random. First, the case where the pmf of N has finite support is considered. Then the result is extended to the case where the pmf of N has infinite support.

3.1. Finite Case

Theorem 6. *The optimal expected accumulated reward EV_1^C can be computed using the recursion*

$$\begin{aligned}EV_j^c=&\frac{\sum_{i=j}^{N_{\max}}P_i}{\sum_{i=j-1}^{N_{\max}}P_i}\\ &\cdot\big[P(W_j\leqslant c, R_j+EV_{j+1}^{c-W_j}\geqslant EV_{j+1}^c)\\ &\quad\times \mathrm{E}\big[R_j+EV_{j+1}^{c-W_j}\,|\,W_j\leqslant c, R_j+EV_{j+1}^{c-W_j}\geqslant EV_{j+1}^c\big]\\ &\quad+[P(R_j+EV_{j+1}^{c-W_j}<EV_{j+1}^c, W_j\leqslant c)\\ &\qquad+P(W_j>c)]EV_{j+1}^c\big],\end{aligned}\tag{4}$$

with boundary conditions $EV_j^c=0$ for any c and $j\geqslant N_{\max}$.

Proof. See e-companion.

3.2. Infinite Case

The result of Theorem 5 can be extended to the case where the pmf for the number of jobs in DSKP has infinite support. For any $j=1,2,\ldots$, $c\in[0,C]$, and $N_{\max}=1,2,\ldots$, let $EV_j^c(N_{\max})$ denote the optimal *conditional* expected accumulated reward from the allocation of resource capacity c to jobs $j,j+1,\ldots,N_{\max}$, given that job $j-1$ has arrived, in the DSKP with the pmf of the number of jobs $P_i^{N_{\max}}=P_i/\sum_{k=1}^{N_{\max}}P_k$ for any $i=1,2,\ldots,N_{\max}$.

Theorem 7. *Assume that $B\equiv \mathrm{E}[\sup_j|X_j/W_j|]<+\infty$, and $P[N<+\infty]=1$. Then for any $j=1,2,\ldots$ and $c\in[0,C]$, the infinite sequence $\{EV_j^c(N_{\max})\}_{N_{\max}=1}^{+\infty}$ converges to the finite limit $EV_j^c(\infty)\equiv\lim_{N_{\max}\to+\infty}EV_j^c(N_{\max})$, and Theorem 5 establishes an optimal policy for DSKP, where the pmf of the number of jobs has infinite support, with EV_j^c replaced by $EV_j^c(\infty)$, $j=1,2,\ldots$, $c\in[0,C]$.*

Proof. See e-companion.

Theorem 7 establishes that an optimal policy for DSKP, where the pmf of the number of jobs has infinite support, can be obtained by sequentially solving a sequence of DSKPs with finite support. First, consider only two jobs, then three, and so on. Then evaluate the limits $\lim_{N_{\max}\to+\infty}EV_j^c(N_{\max})$, $j=1,2,\ldots$, $c\in[0,C]$. Finally, apply Theorem 5 to establish an optimal resource allocation policy.

4. Conclusion

This paper analyzes SSAP and DSKP under the assumption that the number of arriving jobs is random and follows a given discrete distribution. Optimal assignment policies with proofs are provided. Conditioning arguments are key to the solutions to both problems. Note that the complexity of the proposed algorithms is the same as the complexity of the original algorithms introduced by Derman et al. (1972) and Papastavrou et al. (1996).

Note that DSKP, where the pmf of the number of jobs has infinite support, can be solved by alternative methods such as a total reward Markov decision process. Further research is required to assess and compare the performance of these methods. Other challenges include discrete sequential assignment problems in which job values are dependent on each other and/or the workers to whom the jobs are assigned. Also, the proposed models assume that the sequences in which the jobs with their respective value cdfs arrive are fixed and known. Identifying optimal resource allocation policies for the cases in which such sequences could be random is another hard yet important problem.

5. Electronic Companion

An electronic companion to this paper is available as part of the online version that can be found at http://or.journal.informs.org/.

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References

Albright, S. C. 1974. Optimal sequential assignments with random arrival times. *Management Sci.* **21**(1) 60–67.

Derman, C., G. J. Lieberman, S. M. Ross. 1972. A sequential stochastic assignment problem. *Management Sci.* **18**(7) 349–355.

Derman, C., G. J. Lieberman, S. M. Ross. 1975. A stochastic sequential allocation model. *Oper. Res.* **23**(6) 1120–1130.

Kennedy, D. P. 1986. Optimal sequential assignment. *Math. Oper. Res.* **11**(4) 619–626.

Kleywegt, A. J., J. D. Papastavrou. 1998. The dynamic and stochastic knapsack problem. *Oper. Res.* **46**(1) 17–35.

Kleywegt, A. J., J. D. Papastavrou. 2001. The dynamic and stochastic knapsack problem with random sized items. *Oper. Res.* **49**(1) 26–41.

Lu, L. L., S. Y. Chiu, L. A. Cox Jr. 1999. Optimal project selection: Stochastic knapsack with finite time horizon. *J. Oper. Res. Soc.* **50**(6) 645–650.

McLay, L. A., S. H. Jacobson, A. G. Nikolaev. 2009. A sequential stochastic passenger screening problem for aviation security. *IIE Trans.* **41**(6) 575–591.

Nakai, T. 1986. A sequential stochastic assignment problem in a partially observable Markov chain. *Math. Oper. Res.* **11**(2) 230–240.

Nikolaev, A. G., S. H. Jacobson, L. A. McLay. 2007. A sequential stochastic security system design problem for aviation security. *Transportation Sci.* **41**(2) 182–194.

Papastavrou, J. D., S. Rajagopalan, A. J. Kleywegt. 1996. The dynamic and stochastic knapsack problem with deadlines. *Management Sci.* **42**(12) 1706–1718.

Prastakos, G. P. 1983. Optimal sequential investment decisions under conditions of uncertainty. *Management Sci.* **29**(1) 118–134.

Righter, R. L. 1987. The stochastic sequential assignment problem with random deadlines. *Probab. Engrg. Inform. Sci.* **1**(2) 189–202.

Sakaguchi, M. 1972. A sequential assignment problem for randomly arriving jobs. *Rep. Statist. Appl. Res.* **19** 99–109.

Sakaguchi, M. 1983. A sequential stochastic assignment problem with an unknown number of jobs. *Mathematika Japonica* **29**(2) 141–152.

Su, X., S. A. Zenios. 2005. Patient choice in kidney allocation: A sequential stochastic assignment model. *Oper. Res.* **53**(3) 443–455.

Van Slyke, R., Y. Young. 2000. Finite horizon stochastic knapsacks with applications to yield management. *Oper. Res.* **48**(1) 155–172.